Math 2X03 - Homework 5

Due: June 09, 2016

Chapters Covered: Chapter 16.3, 16.4, 16.5, 16.6

1. (Chapter 16.3 # 18) Let $\mathbf{F}(x, y, z) = \sin y\mathbf{i} + (x\cos y + \cos z)\mathbf{j} - y\sin z\mathbf{k}$.
   (a) Find a function f such that $\mathbf{F} = \nabla f$.
   (b) Use your answer in part a) such to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C given by $\mathbf{r}(t) = \sin t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$, $0 \le t \le \frac{\pi}{2}$.

2. (Chapter 16.3 # 19) Show that the line integral is independent of path and evaluate the integral

$$\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$$

   where C is any path from $(1, 0)$ to $(2, 1)$.

3. Use Green's Theorem to evaluate the line integral

$$\int_C xy^2\,dx + 2xy\,dy$$

   along the positively oriented curve C which is the triangle with vertices $(0, 0), (2, 2), (2, 4)$.

4. Let $\mathbf{F}(x, y) = \dfrac{2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j}}{(x^2 + y^2)^2}$. Find

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

   for **any** positively oriented simple closed curve C that encloses the origin.

5. Recall that the Green's Theorem gives us the following formulas for the area of a region D, bounded by the curve C:

$$A(D) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

   Use one of the above formulas to find the area under one arch of the cycloid $x = t - \sin t, y = 1 - \cos t$.

6. (Chapter 16.5 # 8) Find the curl and the divergence of $\mathbf{F} = \left\langle \frac{x}{y}, \frac{y}{z}, \frac{z}{x} \right\rangle$.

7. (a) If f is a scalar field, and $\mathbf{F}$ is a vector field, then $(f\mathbf{F})(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$. Show that $\operatorname{div}(f\mathbf{F}) = f \operatorname{div}\mathbf{F} + \mathbf{F} \cdot \nabla f$.
   (b) If f is a function of three variables, show that

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

   We denote $\operatorname{div}(\nabla f)$ by $\nabla^2 f$. The operator $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace operator**.

(c) We can also apply the Laplace operator ∇^2 to a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

Show that

$$\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}, \tag{1}$$

assuming that the appropriate partial derivatives exist and are continuous.

8. (a) (Chapter 16.5 #33) Use the Green's theorem in the fom

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x,y)\,dA$$

to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\,dA = \oint_C f(\nabla g)\cdot\mathbf{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$$

where D and C satisfy the hypothesis of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g\cdot\mathbf{n} = D_{\mathbf{n}}g$ occurs in the line integral, This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g).

Hint: You will need to use the identity from question 7a

(b) (Chapter 16.5 #34) Use Green's first identity (from above) to prove **Green's second identity**:

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

(c) (Chapter 16.5 #35) A function g is called **harmonic** on D if $\nabla^2 g = 0$ on D(This is called the Laplace's equation). Use the Green's first identity (with the same hypothesis in question 8a to show that if g is harmonic on D, then

$$\oint_C D_{\mathbf{n}}g\,ds = 0,$$

where $D_{\mathbf{n}}g$ is the normal derivative of g defined above. Hint: Apply Green's first identity with $f(x,y) = 1$.

(d) (Chapter 16.5 #36) Use Green's first identity to show that if f is harmonic and D, and if $f(x,y) = 0$ on the boundary curve C, then

$$\iint_D |\nabla f|^2\,dA = 0$$

Assume the same hypothesis as question 8a.

9. (Section 15.5 # 38) Maxwell's equations relating the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \quad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$ (Hint: Use equation 1)

(d) $\nabla^2 \mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$

10. (**Conservation of Energy**) A continuous force field $\mathbf{F}$ moves an object along a path C given by $\mathbf{r}(t)$, $a \leq t \leq b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point. According to *Newton's Second Law of Motion*, the force $\mathbf{F}(\mathbf{r}(t))$ at a point C is related to the acceleration $\mathbf{a} = \mathbf{r}''$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

(a) Show that the work done by the force on the object is

$$W = \frac{m}{2}\left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2\right) \tag{2}$$

The quantity $\frac{m}{2}\left(|\mathbf{v}(t)|^2\right)$ is called the **kinetic energy** of the object, $\mathbf{v} = \mathbf{r}'$ is the velocity.

(b) Use equation (2) to show to show that

$$W = K(B) - K(A) \tag{3}$$

i.e. the work done by a force field along C is equal to the change in kinetic energy at the endpoints of C.

(c) Now let's assume that $\mathbf{F}$ is a conservative force field i.e. $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so $F = -\nabla P$. Show that

$$W = P(A) - P(B) \tag{4}$$

Comparing equation (3) and (4) we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under a **conservative** force field, then the sum of its potential energy and its kinetic energy is constant. This is called the **Law of Conservation of Energy** and this is why the vector field is called *conservative*

11. Find a parametric representation for the following surfaces:

(a) (Chapter 16.6 #21) The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz-plane

(b) (Chapter 16.6 #26) The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

12. (Chapter 16.6 # 38) Find an equation of the tangent plane to the parametric surface $\mathbf{r}(u, v) = (1 - u^2 - v^2)\mathbf{i} - v\mathbf{j} - u\mathbf{k}$; at the point $(-1, -1, -1)$.